Non-Standard (Flag-dipoles) Spinor Fields in $f(R)$ Cosmology

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The Dirac equation in a $f(R)$ Riemann-Cartan cosmological scenario admits solutions [1], that are shown here not to be Dirac spinor fields, which seems to play a fundamental role on the matter fields properties in such cosmological framework, as: a) spinor fields in torsional $f(R)$ or conformal gravity, that satisfy the Dirac equation, correspond to non-standard, flag-pole, spinor fields (in both Lounesto and Wigner classification); b) the field equations imply a constraint among the bilinear covariants associated to the spinor field. Some constraints imposed on the metric and/or on the spinor field imply that an isotropic universe with matter fields circumvent the question that Dirac fields do not undergo the cosmological principle. Remarkably, if the spinor fields are assumed to satisfy the Dirac equation, they are shown not to be legitimate Dirac spinor fields, under Lounesto spinor field classification, but indeed flag-dipole singular spinor fields. Such type of spinor fields can still be led to flagpole spinor fields, which encompass Elko — a prime candidate to the dark matter problem — and Majorana spinor fields. This case is evinced when the spin direction associated to the spinor field vanishes, providing an anisotropic universe without fermionic torsional interactions.

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I. INTRODUCTION

From a purely mathematical perspective, torsion is as necessary and important as the curvature of the spacetime itself. In a geometry that incorporates a differential structure, the introduction of covariant derivatives is as inevitable as the definition of the metric, respectively encoded in terms of the connection and the metric tensor. Moreover, the connection in the most general case is not symmetric as well as the metric is not flat, giving rise to torsion and curvature.

On the other hand, also from a genuine physical point of view, torsion and curvature are both relevant. In fact, according to the Wigner classification of particles in terms of their masses and spin, physical fields are known to be characterized by both energy and spin density. In the most general case, wherein all geometrical quantities must be coupled to corresponding physical fields through specific field equations, the fact that torsion couples to the spin density through the Sciama-Kibble field equations has much the same meaning of the fact that curvature couples to the energy density through the Einstein field equations. The vanishing of torsion is not the most general case but solely the most general spinless situation, in the sense that Einstein gravity is not the most general theory but merely the most general dynamical solution in absence of spinning matter of the Einstein-Sciama-Kibble (ESK) gravity.

Hence the most general ESK theory must be considered when general dynamical matter fields are taken into account. This is clear when filling the spacetime with spin matter fields. In this case the torsion is linked to the spin density of the matter field and, as the theory is formulated at the least-order derivative in all field equations, torsion is algebraically related to the spin density. This has a prominent and interesting consequence: in all curvatures and covariant derivatives, it is possible to split all torsional contributions apart, therefore substituting torsion by the spin density of the spin matter field. As a consequence all field equations of the ESK theory reduce to the same equations that one come across with in the torsionless theory complemented by spin-spin self-interacting potentials, enriching the dynamics. Such an enrichment is so obvious and important that non-linearities appear, even in the matter field equations.

This fact is best realized in the simplest matter field, namely the spinor field satisfying the Dirac equation. Notice that we refrain ourselves to refer to this spinor field as the Dirac spinor field. As

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we shall see, a spinor field is not completely determined by its dynamics: a spinor field obeying the Dirac equation is not necessarily a Dirac spinor field. Dirac fields have torsional effects impelling the $\overline{\phi}\gamma^{\mu}\phi\overline{\phi}\gamma_{\mu}\phi$ potential, which is the usual Nambu-Jona-Lasinio (NJL) potential. It is not unexpected, since one may simply assert that torsion for the Dirac field equation is the geometrical justification of the the NJL potential in the Dirac field equation. Notwithstanding, when one takes a closer look at that potential, it is possible to realize that there is a fundamental difference between torsional effects and the original NJL potential. Indeed, the two potentials appear with opposite sign. It means that torsion is repulsive whereas the NJL potential is attractive. This is expected, since torsion is related to the spin and therefore it must present the character of a centrifugal barrier, which is repulsive, while the NJL potential should not. It was chosen to be attractive for reasons that are here irrelevant. The fact that torsion has rotational features that create a sort of centrifugal barrier has the consequence that the spreading of the wave packet associated to the spinor field increases, and therefore its stability is compromised.

In order to circumvent this problem, one can accept the fact that these fields possess axially symmetric solutions exclusively. Another way consists in the option for which restrictions should be imposed on the spinor field. For instance, by assuming constraints on the form of the spinor so that $\overline{\phi}\gamma^{\mu}\phi\overline{\phi}\gamma_{\mu}\phi = 0$, in order to have no torsional influence on the spinor field dynamics. However, the cognizable risk is to restrict so severely the spinor field that it might degenerate into something that is no longer what it is known to be the Dirac spinor field. In fact, according to the Lounesto spinor field classification, a given spinor field can be cast into six distinct classes: Dirac fields in the proper sense belong to any of the first three classes. Nevertheless, *according to the aforementioned restrictions, one gets a spinor field belonging to the fourth class*, a type-(4) (flag-dipole) spinor field. Corresponding to the fifth (flagpoles) and sixth classes are yet other forms of spinor fields, in particular Majorana, Elko, and Weyl spinor fields. Inasmuch as subsets of the first three classes — as the legitimate Dirac spinor fields — are prominently relevant and well-studied in quantum field theory and its phenomenology, the fourth spinor fields class should be better understood. It contains spinor fields precluding torsion to affect the dynamics provided by the field equations. Herewith, it allows type-(4) spinor fields to be compatible to conditions of stability and spherical symmetry of the background. Therefore, these type-(4) flag-dipole spinor fields would also be compatible with even more symmetric configurations such as those met in cosmological models.

Accordingly, it is essential to better comprise the properties of type-(4) spinor fields, for instance by finding specific solutions in the case of special symmetries in the ESK gravity. However, this may not be enough: the ESK gravity is built on an action presenting the Ricci scalar R alone. Recently

some generalizations emulate the ESK gravity, erected on actions that are generic functions of the Ricci scalar as $f(R)$. They attract much interest, especially in cosmological applications, and thereupon an even stronger result is to find possible solutions for the type-(4) spinor fields dynamics, in the case of ESK-like $f(R)$ gravity.

In this paper one of the main aims is to show that matter fields which are solutions of the Dirac equation, in the cosmological context throughout the paper, are not Dirac spinor fields. We intend to exhibit a physical solution of the Dirac equation that for the first time, as far as we know, is a flag-dipole singular spinor field. Flagpoles and flag-dipoles — type-(5) and -(4) spinor fields under Lounesto classification — are usually disseminated in the literature as a mathematical apparatus to support Penrose flags [2] on the light-cone and twistor theory, among other interesting applications. Although they may be constructed via pure (Weyl) spinor fields in lower dimensions, up to now there was no dynamical equation which matter field solutions corresponded to flag-dipoles. It might seem that trivial geometries are more leaning on the existence of spinor fields other than flag-dipoles. In particular, we shall demonstrate that the Dirac equation in torsional $f(R)$ gravity has flag-dipole spinor fields as solutions.

This paper is organized as follows: in the next Section we introduce the Lounesto classification program according to the bilinear covariants and provide some necessary concepts concerning type-(4) spinor fields. In Sec. III we study the solutions for spinor fields in the context of Riemann-Cartan geometries to the $f(R)$ and conformal gravity cases and show that they are non-standard singular classes of Lounesto's classification. In Sec. IV we conclude. In the Appendix we show how to construct the most general type-(4).

II. NON-STANDARD (FLAG-DIPOLE) SPINOR FIELDS

This Section is devoted to briefly provide some properties on the flag-dipole spinor fields, where the most relevant general properties regarding such spinor fields, and the notation fixed throughout the text as well, are introduced.

Classical spinor fields carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of the Lorentz group $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}^e_{1,3}$. They are sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, where ρ denotes the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C})$ in $\mathbb{C}^4$. Furthermore, classical spinor fields are also sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\rho'} \mathbb{C}^2$, where ρ' is the $D^{(1/2,0)}$ or the $D^{(0,1/2)}$ representation

of SL$(2,\mathbb{C})$ in $\mathbb{C}^2$. Given a spinor field ψ, the bilinear covariants are defined as:

$$\sigma = \psi^\dagger\gamma_0\psi, \quad \mathbf{J} = J_\mu\theta^\mu = \psi^\dagger\gamma_0\gamma_\mu\psi\theta^\mu, \quad \mathbf{S} = S_{\mu\nu}\theta^{\mu\nu} = \frac{1}{2}\psi^\dagger\gamma_0 i\gamma_{\mu\nu}\psi\theta^\mu\wedge\theta^\nu,$$
$$\mathbf{K} = K_\mu\theta^\mu = \psi^\dagger\gamma_0 i\gamma_{0123}\gamma_\mu\psi\theta^\mu, \quad \omega = -\psi^\dagger\gamma_0\gamma_{0123}\psi. \tag{1}$$

The set $\{\gamma_\mu\}$ denote to the Dirac matrices, and the objects in (1) satisfy the so-called Fierz identities [3–5]

$$\mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{J}\llcorner\mathbf{K} = 0 \quad \mathbf{K}^2 = -\mathbf{J}^2, \quad \mathbf{J}\wedge\mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}. \tag{2}$$

A spinor field such that at least one of the ω and the σ are null [not null] is said to be singular [regular]. Lounesto spinor classification is given by the following spinor field classes [3], where $\mathbf{J}$, $\mathbf{S}$, and $\mathbf{K} \neq 0$ for regular spinor fields:

$$1) \qquad \sigma \neq 0, \quad \omega \neq 0. \tag{3}$$
$$2) \qquad \sigma \neq 0, \quad \omega = 0. \tag{4}$$
$$3) \qquad \sigma = 0, \quad \omega \neq 0. \tag{5}$$
$$4) \quad \sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \tag{6}$$
$$5) \quad \sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0. \tag{7}$$
$$6) \quad \sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0. \tag{8}$$

Types-(1), -(2), and -(3) are Dirac spinor fields in the Lounesto classification, and types-(4), -(5), and -(6) are respectively called flag-dipole, flagpole, and Weyl spinor fields. Majorana spinor fields are a particular case of type-(5) spinor fields 7. For Dirac spinor fields, $\mathbf{S}$ is the distribution of intrinsic angular momentum; $\mathbf{J}$ is associated with the current of probability, and $\mathbf{K}$ is associated with the direction of the electron spin [3–5]. The density $\mathbf{J}$ is not zero for any kind of spinor field. For singular spinor fields, $\mathbf{J}^2 = -\mathbf{K}^2 = 0$.

By introducing the element $Z = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}$, Z is denominated a boomerang whenever it satisfies $\gamma_0 Z^\dagger\gamma_0 = Z$. When a spinor field is singular, namely it satisfies $\sigma = 0 = \omega$, the Fierz identities are substituted by the most general identities [4]:

$$Z^2 = \sigma Z, \qquad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \qquad Z\gamma_\mu Z = 4J_\mu Z,$$
$$Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z, \qquad Z\gamma_{0123}Z = -4\omega Z. \tag{9}$$

When one considers a type-(4) flag-dipole spinor fields, the boomerang can be expressed as $Z = \mathbf{J} + i\mathbf{J}s - ih\gamma_{0123}\mathbf{J}$, where s is a spacelike vector orthogonal to $\mathbf{J}$. The distribution of intrinsic

angular momentum $\mathbf{S}$ in (1) is provided by $\mathbf{S} = \mathbf{J} \wedge s$. The real number $h \neq 0$ is defined in order to relate $\mathbf{K}$ and $\mathbf{J}$, as $\mathbf{K} = h\mathbf{J}$, which evinces the definition of helicity, and satisfies $h^2 = 1 + s^2$ [6]. It implies the definition of helicity in quantum mechanics and an equivalent relation for anti-particles as well [7].

Type-(5) spinor fields are obtained as a particular case where $h = 0$. In fact, by the expression $\mathbf{K} = h\mathbf{J}$, when $h = 0$ the expressions $\omega = 0 = \sigma$, $\mathbf{K} = 0$, $\mathbf{J} \neq 0$ hold. Type-(5) spinor fields are therefore limiting cases of type-(4) spinor fields, as well as type-(6) Weyl spinor fields further are. Indeed, it occurs in the limit $s \to 0$, implying that $h = \pm 1$. More details on the most general form of type-(4) spinor fields are provided in the Appendix.

III. MATTER FIELDS IN RIEMANN-CARTAN GEOMETRIES

In the previous Section some features related to type-(4) spinor fields have been introduced, and now we shall take into account the Wigner classification, to further study the spinor fields properties. According to the Wigner classification in terms of irreducible representations of the Poincaré group, quantum particles are classified in terms of their mass and spin labels. The corresponding quantities for the quantum fields are given in terms of energy and spin densities. If one wishes to pursue the same spirit Einstein followed in developing gravity, expressing the field equations by coupling the curvature to energy in the most general case where torsion is present, one is compelled to write similar field equations coupling torsion to the spin. When this is accomplished in the most straightforward way, the Einstein equations for the curvature-energy coupling are generalized as to include the Sciama-Kibble equations for the torsion-spin coupling. Namely, the ESK system of field equations, which can be obtained by generalizing the Ricci scalar written in terms of the metric $R(g)$ by the Ricci scalar written in terms of both metric and torsion $R(g, T)$ in the action, and subsequently varying it with respect to the two independent fields.

Notwithstanding, this is merely the most straightforward generalization of gravity with torsion. Other more general theories can be obtained by adding torsion not only implicitly through the curvature, but explicitly as well, as quadratic terms beside the curvature $R(g, T) + T^2$ in the action. Once the field equations are written down, and all torsional contributions are separated and converted into spinor interactions, the effects of these extensions are reduced to a simple scaling of the torsional terms, or equivalently of the spinor interaction. It is evinced by introducing new coupling constants for such spin potentials. This is cogent in itself, because one of the most important problems about torsion in gravity, namely the fact that torsion should have been relevant

only at the Planck scales, can be overcome. In fact, in these theories torsion has its own coupling constant, that does not necessarily coincide with the gravitational constant [8, 9]. On the other hand however, those theories do not encompass the possibility to have *dynamical* extensions, such as those provided by higher-order derivative field equations. The two most important higher-order derivative theories are the one for which the Ricci scalar R is replaced by a generic function $f(R)$ in the action [10], and the one that is capable of implementing the conformal symmetry in the action itself [11, 12]. In the following we shall deal with both of them.

A. Torsional $f(R)$-Gravity

This extension of the Einstein-Hilbert action to be a generic function $f(R)$ is captivating, since it is the most general, whenever one restricts the Ricci scalar as the sole source of dynamical information. In the case which in both metric and torsion are taken into account, the variation with respect to a general metric g and a g-compatible connection Γ, or equivalently a tetrad field e and a spin-connection ω, yields the metric-affine or tetrad-affine approaches [13–16]. The correspondent field equations are

$$T_{ij}{}^{h} = \frac{1}{f'(R)}\left[\frac{1}{2}\left(\frac{\partial f'(R)}{\partial x^p} + S_{pq}{}^{q}\right)\epsilon_r{}^{ph}\epsilon_i^{r}{}_j + S_{ij}{}^{h}\right], \tag{10a}$$

$$\Sigma_{ij} = f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij}, \tag{10b}$$

where R_{ij}, ϵ_{ijk}, and $T_{ij}{}^{h}$ are the Ricci, the Levi-Civita, and the torsion tensors respectively. The Σ_{ij} and $S_{ij}{}^{h}$ denote the stress-energy and spin density tensors associated to the matter fields: the conservation laws

$$\nabla_i\Sigma^{ij} + T_i\Sigma^{ij} - \Sigma_{pi}T^{jpi} - \frac{1}{2}S_{sti}R^{stij} = 0, \tag{11a}$$

$$\nabla_h S^{ijh} + T_h S^{ijh} + \Sigma^{ij} - \Sigma^{ji} = 0, \tag{11b}$$

follow from the Bianchi identities [10]. In Eqs.(11) the symbols ∇_i and R^{ijkl} denote respectively the covariant derivative and the curvature tensor, with respect to the dynamical connection Γ. By denoting $\Gamma^i = e^i_\mu\gamma^\mu$, where e^μ_i is a tetrad associated with the metric, and by introducing $\mathrm{S}_{\mu\nu} := \frac{1}{8}[\gamma_\mu, \gamma_\nu]$ the covariant derivative of the matter field ψ is denoted by $D_i\psi = \frac{\partial\psi}{\partial x^i} + \omega_i{}^{\mu\nu}\mathrm{S}_{\mu\nu}\psi$ and $D_i\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^i} - \bar{\psi}\omega_i{}^{\mu\nu}\mathrm{S}_{\mu\nu}$, where $\omega_i{}^{\mu\nu}$ is the spin connection. One can furthermore indite $D_i\psi = \frac{\partial\psi}{\partial x^i} - \Omega_i\psi$ and $D_i\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^i} + \bar{\psi}\Omega_i$ where

$$\Omega_i := -\frac{1}{4}g_{jh}\left(\Gamma_{ik}{}^{j} - e^j_\mu\partial_i e^\mu_k\right)\Gamma^h\Gamma^k. \tag{12}$$

${\Gamma_{ik}}^j$ denote the coefficients of the linear connection Γ, since the relation between linear and spin connection is provided by ${\Gamma_{ij}}^h = {\omega_i}^\mu{}_\nu e^h_\mu e^\nu_j + e^h_\mu \partial_i e^\mu_j$, as can be immediately calculated. In the case of matter fields, the spin density tensor is given by ${\mathrm{S}_{ij}}^h = \frac{i}{2}\bar{\psi}\left\{\Gamma^h, \mathrm{S}_{ij}\right\}\psi \equiv -\frac{1}{4}\eta^{\mu\sigma}\epsilon_{\sigma\nu\lambda\tau}K^\tau e^h_\mu e^\nu_i e^\lambda_j$. Remember that K^τ is the component of the pseudovector bilinear covariant defined at (1). The stress-energy tensor components of the matter fields are hence described as

$$\Sigma^D_{ij} := \frac{i}{4}\left(\bar{\psi}\Gamma_i D_j\psi - D_j\bar{\psi}\Gamma_i\psi\right) \qquad \text{and} \qquad \Sigma^F_{ij} := (\rho + p)U_iU_j - pg_{ij}. \tag{13}$$

In Eqs.(13), ρ, p and U_i denote respectively the matter-energy density, the pressure, and the four-velocity of the fluid. The trace of the equations (10b)

$$f'(R)R - 2f(R) = \Sigma, \tag{14}$$

is supposed to relate the Ricci scalar curvature R and the trace Σ of the stress-energy tensor, as in [10, 13–15]. Furthermore, it is assumed that $f(R) \neq kR^2$ — since the case $f(R) = kR^2$ is solely compatible with the condition $\Sigma = 0$. Now, from Eq.(14) it is possible to express $R = F(\Sigma)$, where F is an arbitrary function. Furthermore, introducing the scalar field $\varphi := f'(F(\Sigma))$ as well as the effective potential $V(\varphi) := \frac{1}{4}\left[\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2(f')^{-1}(\varphi)\right]$, the field equations (10b) are written in the Einstein-like form

$$\begin{aligned}\mathring{R}_{ij} - \frac{1}{2}\mathring{R}g_{ij} = \frac{1}{\varphi}\Sigma^F_{ij} + \frac{1}{\varphi}\Sigma^D_{ij} + \frac{1}{\varphi^2}\bigg(&-\frac{3}{2}\varphi_i\varphi_j + \varphi\mathring{\nabla}_j\varphi_i + \frac{3}{4}\varphi_h\varphi_k g^{hk}g_{ij} \\ &-\varphi\mathring{\nabla}^h\varphi_h g_{ij} - V(\varphi)g_{ij}\bigg) + \mathring{\nabla}_h{\hat{\mathrm{S}}_{ji}}{}^h + {\hat{\mathrm{S}}_{hi}}{}^p{\hat{\mathrm{S}}_{jp}}{}^h - \frac{1}{2}{\hat{\mathrm{S}}_{hq}}{}^p{\hat{\mathrm{S}}^q}{}_p{}^h g_{ij},\end{aligned} \tag{15}$$

where $\mathring{R}_{ij}$, $\mathring{R}$ and $\mathring{\nabla}_i$ denote respectively the Ricci tensor, the Ricci scalar curvature and the covariant derivative of the Levi–Civita connection. ${\hat{\mathrm{S}}_{ij}}{}^h$ and φ_i denote ${\hat{\mathrm{S}}_{ij}}{}^h := -\frac{1}{2\varphi}{\mathrm{S}_{ij}}{}^h$ and $\varphi_i := \frac{\partial\varphi}{\partial x^i}$, respectively. In addition to this, the generalized Dirac equations for the spinor field are in this context

$$i\Gamma^h D_h\psi + \frac{i}{2}T_h\Gamma^h\psi - m\psi = 0, \tag{16}$$

where $T_h := {T_{hj}}^j$ is the axial torsion[1]. The symmetrized part of the Einstein-like equations (15) as well as the Dirac equations (16) are written as [10]

$$\begin{aligned}\mathring{R}_{ij} - \frac{1}{2}\mathring{R}g_{ij} = \frac{1}{\varphi}\Sigma^F_{ij} + \frac{1}{\varphi}\mathring{\Sigma}^D_{ij} + \frac{1}{\varphi^2}\bigg(&-\frac{3}{2}\varphi_i\varphi_j + \varphi\mathring{\nabla}_j\varphi_i + \frac{3}{4}\varphi_h\varphi_k g^{hk}g_{ij} \\ &-\varphi\mathring{\nabla}^h\varphi_h g_{ij} - V(\varphi)g_{ij}\bigg) + \frac{3}{64\varphi^2}K^\tau K_\tau g_{ij}\end{aligned} \tag{17}$$

[1] It is interesting to note that at this point it is not formally explicit by (16) whether we are dealing with Dirac equation with torsion in a simply connected space or with a Dirac equation without torsion in a multiply connected space-time [17]. As both descriptions are mathematically equivalent, we can transpose one formalism into another, in order to circumvent such question.

and

$$i\Gamma^h \mathring{D}_h \psi - \frac{3}{16\varphi}\left[\sigma + i\omega\gamma_5\right]\psi - m\psi = 0, \tag{18}$$

where $\mathring{\Sigma}^D_{ij} := \frac{i}{4}\left[\bar{\psi}\Gamma_{(i}\mathring{D}_{j)}\psi - \left(\mathring{D}_{(j}\bar{\psi}\right)\Gamma_{i)}\psi\right]$ and $\mathring{D}_i$ is the covariant derivative of the Levi–Civita connection.

As spinor fields satisfying the Dirac equation in this scenario are incompatible with stationary spherical symmetry [18], the simplest choice for the background must be at least an axially symmetric Bianchi-I type metric, given by the form $ds^2 = dt^2 - a^2(t)\,dx^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2$, where the $\Gamma^i = e^i_\mu \gamma^\mu$ are given by

$$\Gamma^0 = \gamma^0, \qquad \Gamma^1 = \frac{1}{a(t)}\gamma^1, \qquad \Gamma^2 = \frac{1}{b(t)}\gamma^2, \qquad \Gamma^3 = \frac{1}{c(t)}\gamma^3, \tag{19}$$

and the tetrad field is given by $e^\mu_0 = \delta^\mu_0$, $e^\mu_1 = a(t)\delta^\mu_1$, $e^\mu_2 = b(t)\delta^\mu_2$, and $e^\mu_3 = c(t)\delta^\mu_3$, for $\mu = 0,1,2,3$. The spin-Dirac operator acts on spinor fields and their conjugates respectively as $\mathring{D}_i\psi = \partial_i\psi - \mathring{\Omega}_i\psi$ and $\mathring{D}_i\bar{\psi} = \partial_i\bar{\psi} + \bar{\psi}\mathring{\Omega}_i$, where the spin connection coefficients $\mathring{\Omega}_i$ are given by (introducing the notation $a_1 = a, a_2 = b, a_3 = c$)

$$\mathring{\Omega}_0 = 0, \qquad\qquad \mathring{\Omega}_i = \frac{1}{2}\dot{a}_i\gamma^i\gamma^0.$$

Therefore, the Einstein-like equation (17) reads

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} = \frac{\rho}{\varphi} - \frac{3}{64\varphi^2}K^\sigma K_\sigma + \frac{1}{\varphi^2}\left[-\frac{3}{4}\dot{\varphi}^2 - \varphi\dot{\varphi}\frac{\dot{\tau}}{\tau} - V(\varphi)\right], \tag{20a}$$

$$\frac{\ddot{a}_r}{a_r} + \frac{\ddot{a}_s}{a_s} + \frac{\dot{a}_r}{a_r}\frac{\dot{a}_s}{a_s} = -\frac{p}{\varphi} + \frac{1}{\varphi^2}\left[\varphi\dot{\varphi}\frac{\dot{a}_t}{a_t} + \frac{3}{4}\dot{\varphi}^2 - \varphi\left(\ddot{\varphi} + \frac{\dot{\tau}}{\tau}\dot{\varphi}\right) - V(\varphi)\right] + \frac{3}{64\varphi^2}K^\sigma K_\sigma, \tag{20b}$$

where r, s, t denote indexes $1,2,3$ different from each other. The Dirac field equation (18) assumes the form

$$\dot{\psi} + \frac{\dot{\tau}}{2\tau}\psi + im\gamma^0\psi - \frac{3i}{16\phi}(\sigma\gamma^0 + i\omega\gamma^0\gamma^5)\psi = 0, \tag{21}$$

where $\tau := abc$ [19, 20]. Together with the conditions

$$\mathring{\Sigma}^D_{rs} = 0 \quad\Rightarrow\quad a_r\dot{a}_s - a_s\dot{a}_r = 0 \quad\cup\quad K^t = 0, \tag{22}$$

the equations $\mathring{\Sigma}^D_{0i} = 0$ are automatically satisfied. Finally, the conservation laws together with an equation of state of the kind $p = \lambda\rho$ — here λ is a number between 0 and 1 — yield $\dot{\rho} + \frac{\dot{\tau}}{\tau}(1+\lambda)\rho = 0$, which completes the whole set of field equations, having the general solution given by

$$\rho = \rho_0\tau^{-(1+\lambda)} \qquad \rho_0 = \text{constant}. \tag{23}$$

The matter field in such axially symmetric background is such that conditions (22) are constraints imposed on the metric or on the matter field. They exist if and only if one of the following conditions hold:

a) imposing constraints of purely geometrical origin by forcing that $a\dot{b} - b\dot{a} = 0$, $a\dot{c} - c\dot{a} = 0$, $c\dot{b} - b\dot{c} = 0$. In this scenario there are fermionic matter fields in an isotropic universe, which might *a priori* cause some problem, due to the fact that it is known that Dirac fields do not undergo the cosmological principle [21]. But the result by Tsamparlis [21], although valid for Dirac spinor fields, does not hold for the other types, according to Lounesto spinor field classification.

b) another condition is to impose constraints of purely material origin by requiring that the spatial components of the spin direction satisfy $K_i = 0$. This represents an anisotropic universe devoid of coupling terms matter/axial torsion. In this case there is no fermionic torsional interactions. Indeed, the particle spin interacts with the axial component of the torsion tensor, and when the spatial components of the spin direction equal zero it implies that such particles described by the field ψ do not interact to the torsion. It was argued in [1] to be unsatisfactory: if Dirac fields are absent then it is not clear what may then justify anisotropies.

c) the last situation would be of both geometrical and material origin by insisting that for instance $a\dot{b} - b\dot{a} = 0$ with $K_1 = 0 = K_2$, giving a partial isotropy for only two axes with the corresponding two components of the spin vector vanishing. It describes a universe shaped as an ellipsoid of rotation about the only axis along which the spin vector does not vanish. Notice that by insisting on the proportionality between two couples of axes we inevitably get the total isotropy of the 3-dimensional space. Therefore, the situation in which we have $a = b$ with $K_1 = 0 = K_2$ is the only one that we believe to be entirely satisfactory, and we shall henceforth work in this situation, where the only spatial component of the spin direction is $K_3 \neq 0$.

Here, the Dirac and Einstein-like equations (21) and (20) can be worked out as in [19, 20]: for instance, through suitable combinations of (20) we obtain the equations

$$\frac{d}{dt}(J_0\tau) = 0 = \frac{d}{dt}(\sigma\tau) + \frac{3}{8\varphi}(\omega K_0\tau), \tag{24a}$$

$$-\frac{d}{dt}(\omega\tau) + \left[2m + \frac{3\sigma}{8\varphi}\right](K_0\tau) = 0 = \frac{d}{dt}(K_0\tau) + 2m(\omega\tau). \tag{24b}$$

while from Eqs.(24) it is straightforward to deduce that

$$(K_3)^2 = \sigma^2 + \omega^2 + (K_0)^2 = \frac{C^2}{\tau^2}, \qquad\qquad (J_0)^2 = \frac{D^2}{\tau^2}, \tag{25}$$

with C and D constants. It is worthwhile to emphasize that in this special case the theory has an additional discrete symmetry provided by the discrete transformation $\psi \mapsto \gamma^5\gamma^0\gamma^1\psi$, under which all field equations are invariant. It implies that in the Dirac equation the total number of four complex components is in this case reduced to two complex components. Such assertion is equivalent to take flagpole spinor fields, that have four real parameters. Hence (24) are the field equations to be solved. The compatibility with all constraints allows only three classes of spinor fields, each of which has a general member written in one of the following form

$$\psi = \frac{1}{\sqrt{2\tau}} \begin{pmatrix} \sqrt{K-C}\cos\zeta_1 e^{i\theta_1} \\ \sqrt{K+C}\cos\zeta_2 e^{i\vartheta_1} \\ \sqrt{K-C}\sin\zeta_1 e^{i\vartheta_2} \\ \sqrt{K+C}\sin\zeta_2 e^{i\theta_2} \end{pmatrix},$$

with constraints $\tan\zeta_1\tan\zeta_2 = (-1)^{n+1}$ and $\theta_1+\theta_2-\vartheta_1-\vartheta_2 = \pi n$ for any n integer, and also

$$\psi = \frac{1}{\sqrt{2\tau}} \begin{pmatrix} \sqrt{K-C}\cos\zeta_1 e^{i\theta_1} \\ 0 \\ 0 \\ \sqrt{K+C}\sin\zeta_2 e^{i\theta_2} \end{pmatrix} \quad \text{and} \quad \psi = \frac{1}{\sqrt{2\tau}} \begin{pmatrix} 0 \\ \sqrt{K+C}\cos\zeta_1 e^{i\vartheta_1} \\ \sqrt{K-C}\sin\zeta_2 e^{i\vartheta_2} \\ 0 \end{pmatrix}. \tag{26}$$

with ζ_1, ζ_2, θ_1, θ_2, ϑ_1, ϑ_2 being time dependent. The most interesting case for us is one of the latter two cases, for instance the second spinor field at (26) as

$$\psi = \frac{1}{\sqrt{2\tau}} \begin{pmatrix} 0 \\ \sqrt{K+C}e^{i\beta(t)} \\ \sqrt{K-C}e^{-i\beta(t)} \\ 0 \end{pmatrix}, \tag{27}$$

for $\beta(t) = -mt - \frac{3C}{16}\int\frac{dt}{\tau}$. There are further constraints $\sigma = \frac{C}{\tau}$, $\psi^\dagger\psi = \frac{K}{\tau}$ and $\omega = 0 = K_0$. Notice that such a spinor field *is a type-(4), flag-dipole, spinor field, according to the Lounesto spinor fields classification* [22]. It is a remarkable fact: once it is assumed a spinor field ψ in a $f(R)$-Riemann-Cartan cosmology, some type-(4) spinor fields are obtained as the spinor fields (26). Indeed, there is no assumption in Eq.(16) that ψ is a legitimate Dirac spinor field: it only regards *a priori* a spinor field ψ that satisfies the Dirac equation. As far as we know, this is, up to now, the unique physical system whose acceptable solution is given in terms of such spinors.

On the other hand, when one imposes $K_3 = 0$ as a constraint of purely material origin, Eqs.(25) implies that $K_0 = 0$. Therefore $K^\mu = 0$ and we obtain a type-(5) spinor field under Lounesto spinor field classification, which encompasses Majorana and Elko dark spinor fields. It must be stressed that the condition $K_3 = 0$ does not necessarily imply that in this case there is no fermion fields satisfying the Dirac equation (16). In fact Majorana spinor fields are well known to satisfy Dirac equation, although Elko spinor fields do not satisfy it[2].

In summary, by the solutions above, the so called (heretofore) Dirac fields ψ in (26, 27) are not a Dirac spinor field according to Lounesto classification, but a type-(4) flag-dipole spinor field. Besides, since $K_i = 0$ and in particular $K_3 = 0$, by (25) it implies that we are concerning now a type-(5) spinor field, which is a flagpole. But in this case, it is well-known that type-(5) encompasses Elko, Majorana, and the complementary spinor fields, presented at ((B8)). Elko, however, is well known not to satisfy Dirac equations, so as we departure from (25), Elko is excluded to be a solution of such system. The point to be stressed here is that according to the Lounesto spinor field classification, ψ can be allocated in any of the six disjoint classes and there is no *ab initio* relation between the type of the spinor field and the associated dynamics. As mentioned, for instance, the types-(1), (2) and (3) are Dirac spinor fields in the Lounesto classification, having some subset satisfying the Dirac equation. By the same token, type-(6) spinor fields encompass Weyl spinor fields, that indeed satisfy Dirac equations. Nevertheless, it was an open problem whether type-(4) spinor fields satisfy or not the Dirac equations, but the Dirac equations is shown to be dynamically forbidden for the solutions found [1].

B. Torsional Conformal-Gravity

It is worth to finalize this paper pinpointing some recent progress in the study of spinor fields in generalized gravity, as well as some open issues which are under current investigation. While it is somewhat apart of the main theme of the paper, it is certainly enriching from the bookkeeping purposes. In this vein, another interesting higher-order theory of gravity is the one with two curvatures, because this is the only case in which conformal invariance can be obtained [11]. As it turns out, there are two ways to implement conformal transformations for torsion: the first is to

[2] In fact, Elko spinor fields [23, 24] are eigenspinors of the charge conjugation operator and do not satisfy Dirac equation [23], establishing that $(i\gamma^\mu D_\mu \pm m\mathbb{I})$ do not annihilate the Elko type-(5) dark spinor fields. Some important applications are provided, for instance, at [25]. There is still the complementary set of Elko and Majorana fields, with respect to the type-(5) spinor, whose dynamics is still unknown. Its general form is provided in the Appendix B.

require the most general (reasonable) conformal transformation for torsion (where by reasonable we mean reasonable according to what is discussed, for instance, in [26]). The another is to insist on the fact that no conformal transformation is to be given to torsion (because conformal transformations are of metric origin while torsion is independent on the metric). In the former case, because conformal transformations link the metric to torsion, one must modify the Riemann curvature with quadratic-trace torsion terms in order to get a curvature whose irreducible part is conformally invariant [11]. In the latter case, torsion and curvature are separated and essentially independent. Consequently, in the former case [11] the field equations are closely intertwined together, while in the latter case the field equations are independent thus maintaining the curvature-energy and torsion-spin coupling in the spirit of the ESK field equations.

1. Torsion with general Conformal Transformations

In the first case the coupling to the Dirac field has been studied in [11]; however, because in this case the field equations that couple torsion to spin are not invertible in general, torsion cannot be substituted by the spin density into the Dirac field equations, which therefore remain of the general form

$$i\gamma^{\mu}\mathring{D}_{\mu}\psi+\tfrac{3}{4}W_{\sigma}\gamma^{5}\gamma^{\sigma}\psi=0, \tag{28}$$

where W_{σ} is the axial vector dual of the completely antisymmetric part of the torsion tensor. What it implies is that the arguments used in [18] cannot be recovered, and therefore stationary spherically symmetric symmetries are possible. However, in such a case, the complete antisymmetry of the Dirac field does not turn into the complete antisymmetry of torsion but rather in constraints for the gravitational fields that cannot be satisfied in general situations. In this case of general torsional conformal transformations the Dirac field appears to be ill-defined.

An alternative situation is therefore to study Elko fields, which has been done in [12]. However, their dynamics in terms of cosmological solutions has not been studied yet.

2. Torsion with no Conformal Transformations

In the last case [11] the coupling to the Dirac field has been studied showing that the complete antisymmetry of the spin density results into the complete antisymmetry of the torsion tensor, whose dual is an axial vector given by

$$W_{\rho}=\left(\tfrac{4a}{\hbar}\,K^{\mu}K_{\mu}\right)^{-1/3}J_{\rho}, \tag{29}$$

so that torsion can be replaced with the spin density of the spinor field, and the Dirac field equation become

$$i\gamma^{\mu}\mathring{D}_{\mu}\psi-\left(\tfrac{256a}{27}K^{\rho}K_{\rho}\right)^{-\frac{1}{3}}\overline{\psi}\gamma_{\nu}\psi\gamma^{\nu}\psi=0, \tag{30}$$

with non-linear self-interaction that are renormalizable; after a simple Fierz rearrangement they can be written as

$$i\gamma^{\mu}\mathring{D}_{\mu}\psi-\left(\tfrac{27}{256a}\right)^{1/3}\left[\sigma^{2}+i\omega^{2}\right]^{-1/3}\left[\sigma\mathbb{I}-\omega\gamma_{5}\right]\psi=0, \tag{31}$$

clearly showing that the type-(4) spinor fields would verify a Dirac field equation of the form

$$i\gamma^{\mu}\mathring{D}_{\mu}\psi=0, \tag{32}$$

as if torsion were never present, precisely like the ESK theory. In this case, it again happens that the reasoning performed in [18] does not apply, stationary spherically symmetric solutions are possible, the gravitational field equations would reduce to the torsionless spherically symmetric Weyl field equations in a Schwarzschild spacetime. For this type of conformal gravity, the case of Elko fields has not been studied yet.

IV. CONCLUSION

In this paper, we have explored both the regular and singular spinor fields, establishing the general gravitational background with torsion in which the spinor fields are supposed to live in. We have proved that some singular flag-dipoles spinor fields are physical solutions for the Dirac equation in torsional $f(R)$-gravity.

Indeed, when considering Dirac-like fields in $f(R)$-gravity, the presence of torsion imposes the use of an anisotropic background in which the geometric side is diagonal, while, because of the intrinsic features of the spinor field, the energy tensor is not diagonal. In this circumstance, the non-diagonal part of the gravitational field equations results into the constraints (22) characterizing the structure of the spacetime, or the helicity of the spinor field, or both. In our understanding, the only physically meaningful situation is the one in which two axes are equal and one spatial component of the axial vector torsion does not vanish. It provides a universe that is spatially shaped as an ellipsoid of rotation revolving about the only axis along which the spin density is not equal to zero. In the case of conformal gravity, except for the case of torsional conformal transformations, for which the Dirac field seems not well-defined, the case of torsion without con

formal transformations appears to be well-posed; in this case, the gravitational background is much like the torsionless one, and singular type-(4) spinor fields may still emerge.

In this sense, matter spinor fields do not necessarily describe regular matter fields in these theories. In other words, the fact that a spinor field satisfies the Dirac field equation does not forbid it to be singular: spinor fields with peculiar properties may always appear.

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Appendix A: Algebraic, Operator and Classical spinor fields, and the most general types-(4) and -(5)

Let V be a finite n-dimensional real vector space and $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ the space of multivectors over V, where $\Lambda^k(V)$ denotes the k-forms vector space. Given $\psi \in \Lambda(V)$, $\hat{\psi} = (-1)^k \psi$. The conjugation is defined as the reversion composed with the graded involution. If V is endowed with a metric η one extends it to the whole $\Lambda(V)$. Given $\tau, \psi, \xi \in \Lambda(V)$, the left contraction is defined implicitly by $\eta(\tau \lrcorner \psi, \xi) = \eta(\psi, \tilde{\tau} \wedge \xi)$. The Clifford product between $\mathbf{v} \in V$ and ψ is provided by $\mathbf{v}\psi = \mathbf{v} \wedge \psi + \mathbf{v} \lrcorner \psi$. The pair $(\Lambda(V), \eta)$ endowed by the Clifford product is the Clifford algebra $\mathcal{C}\ell_{p,q}$ of $V = \mathbb{R}^{p,q}$, $p + q = n$. Hereon the case $\mathbb{R}^{1,3}$ is going to be considered. All spinor fields are placed in a manifold which locally is a Minkowski spacetime $(M, \eta, \mathring{D}, \tau_\eta, \uparrow)$ in what follows, where $M \simeq \mathbb{R}^4$ is a manifold, $\mathring{D}$ denotes the Levi-Civita connection associated with η, M is oriented by the 4-volume element τ_η and time-oriented by $\uparrow$. Furthermore, $\{\mathbf{e}_\mu\}$ $(\mu=0,1,2,3)$ is a section of the frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$. $\{\mathbf{e}^\mu\}$ is the frame such that $\mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$, with $\{\theta^\mu\}$ and $\{\theta_\mu\}$ respectively the dual bases of $\{\mathbf{e}_\mu\}$ and $\{\mathbf{e}^\mu\}$. Furthermore $\mathbf{e}^5 = i\mathbf{e}^0\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3$. Hereupon we denote $\mathbf{e}_{\mu\nu} = \mathbf{e}_\mu\mathbf{e}_\nu$ and $\mathbf{e}_{\mu\nu\rho} = \mathbf{e}_\mu\mathbf{e}_\nu\mathbf{e}_\rho$.

Ideal (algebraic) and operator spinor fields in the spacetime algebra are equivalent concepts [3]. The problem of representing spinor fields by completely skew-symmetric tensor fields (differential forms) comes back to Ivanenko, Landau and Fock in 1928, and was considered several times [3, 27]. An element $\Psi \in \mathcal{C}\ell^+_{1,3}$ in the even Clifford subalgebra — which corresponds to an operator spinor

— can be written as $\Psi = p + p^{\mu\nu}\mathbf{e}_{\mu\nu} + p^{0123}\mathbf{e}_{0123}$. Using, for instance, the standard representation, Ψ can be represented by

$$\begin{pmatrix} p + ip^{21} & -p^{31} + ip^{32} & p^{30} + ip^{0123} & p^{10} - ip^{20} \\ -p^{13} - ip^{23} & p + ip^{12} & -p^{01} - ip^{02} & p^{03} + ip^{0123} \\ -p^{03} + ip^{0123} & -p^{01} + ip^{02} & p - ip^{12} & p^{13} - ip^{23} \\ -p^{01} - ip^{02} & p^{03} + ip^{0123} & -p^{13} - ip^{23} & p + ip^{12} \end{pmatrix} := \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}.$$

Algebraic spinor fields ψ are elements of the minimal left ideal $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$, where, for instance one can take without loss of generality the idempotent $f = \frac{1}{4}(1+\mathbf{e}_5)(1+i\mathbf{e}_{12})$, associated with the Weyl representation. Hence $\psi = \Phi\frac{1}{2}(1 + i\mathbf{e}_{12}) \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$, where $\frac{1}{2}\Phi = \Phi\frac{1}{4}(1 + \mathbf{e}_5)$ is the real part of ψ. Using the matrix representation it follows the equivalence of operator, algebraic, and classical spinor fields

$$(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f \ni \psi \simeq \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \stackrel{\sim}{\cdot} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \in \mathbb{C}^4. \tag{A1}$$

Appendix B: General type-(4) and type-(5) spinor fields

In order to better understand the structure of type-(4) and their limiting case type-(5) spinor fields, the question we would like to answer is: what is the general form of these spinor fields? In order to answer it, let us take a general spinor given by $\psi = (f, g, \eta, \xi)^T$, with $f, g, \eta, \xi \in \mathbb{C}$, and the definition of these spinor types given by Lounesto classification [3].

Type-(4)

In this Subsection we aim to characterize the most general type-(4) flag-dipole spinor field. By taking into account the expressions (6), (7), and (8) for singular spinor fields, the condition $\sigma = 0 = \omega$ results $\eta f^* + \xi g^* = 0$. We have to analyze the possibilities evinced from this equation. If $f = 0 = g$ or $\eta = 0 = \xi$, it implies a type-(6) spinor field, with $\mathbf{S} = 0$, and therefore this possibility must be dismissed here. It remains the conditions: either $\eta = 0 = \xi$, $f = 0 = g$, or none of components can be null. In this last case, one can isolate a part of them, for example $f = \frac{g\eta\xi^*}{\|\eta\|^2}$. Further, the condition $\mathbf{K} \neq 0$ induces the following possibilities:

a) If $\eta = 0 = \xi$, hence $K_1 = K_2 = 0$, and $K_0 \neq 0 \neq K_3 \Rightarrow \|f\|^2 \neq \|\xi\|^2$;

b) If $f = 0 = \xi$ it implies that $K_1 = K_2 = 0$, and $K_0 \neq 0 \neq K_3 \Rightarrow \|g\|^2 \neq \|\eta\|^2$;

c) If all the components are not null, $K_1 \neq 0 \neq K_2 \Rightarrow \|g\|^2 \neq \|\eta\|^2$.

In the third case, if $\|g\|^2 = \|\eta\|^2$, therefore $\mathbf{K} = 0$. Furthermore, still in the third case, $\|g\|^2 \neq \|\eta\|^2 \Leftrightarrow \|f\|^2 \neq \|\xi\|^2$. Thus, the possible type-(4) spinor fields are:

$$\psi_{(4)} = \begin{pmatrix} f \\ 0 \\ 0 \\ \xi \end{pmatrix}, \ \|f\|^2 \neq \|\xi\|^2; \quad \psi_{(4)} = \begin{pmatrix} 0 \\ g \\ \eta \\ 0 \end{pmatrix}, \|g\|^2 \neq \|\eta\|^2; \quad \psi_{(4)} = \begin{pmatrix} \frac{g\eta\xi^*}{\|\eta\|^2} \\ g \\ \eta \\ \xi \end{pmatrix}, \ \|g\|^2 \neq \|\eta\|^2. \tag{B1}$$

If some inequality associated to one of the spinors above does not hold, it turns forthwith to be a type-(5), which shall be analyzed in what follows.

Type-(5)

We start by noticing how the conditions on the bilinear covariants associated to a type-(5) spinor field imply the following conditions on the spinor field components:

$$\sigma = \overline{\psi}\psi = 0 = -\overline{\psi}\gamma_{0123}\psi = \omega \Longrightarrow \eta f^* + \xi g^* = 0, \tag{B2}$$

$$K_1 = \overline{\psi} i\gamma_{0123}\gamma_1\psi = 0 = \overline{\psi} i\gamma_{0123}\gamma_2\psi = K_2 \Longrightarrow g f^* + \xi\eta^* = 0, \tag{B3}$$

$$K_0 = \overline{\psi} i\gamma_{0123}\gamma_0\psi = 0 = \overline{\psi} i\gamma_{0123}\gamma_3\psi = K_3 \Longrightarrow |f|^2 = |\xi|^2 \text{ and } |g|^2 = |\eta|^2. \tag{B4}$$

Eq. (B4) can be obtained from (B2) and (B3), which are therefore essential to characterize type-(5) spinor fields. In this vein, an equation candidate to describe the dynamics of these general spinor fields must keep (B2) and (B3) invariant. Elko spinor fields obey these equations.

By performing a straightforward calculation with the aid of Eqs.(B2) and (B3) it is possible to obtain

$$f = -\xi^*(\eta + g)(\eta^* + g^*)^{-1} = -\xi^* \left[\frac{\eta + g}{\|\eta + g\|}\right]^2, \tag{B5}$$

and by taking $\tan\varphi_1 = -i\frac{\eta+g-(\eta+g)^*}{\eta+g+(\eta+g)^*}$, we can write $f = -\xi^* e^{2i\varphi_1}$ and $g = -\eta^* e^{2i\varphi_2}$, where φ_1 and φ_2 are related by [3]

$$\tan\varphi_2 = -i\frac{\xi(1 + e^{-2i\varphi_1}) - [\xi(1 + e^{-2i\varphi_1})]^*}{\xi(1 - e^{-2i\varphi_1}) + [\xi(1 - e^{-2i\varphi_1})]^*} = -\cot\varphi_1. \tag{B6}$$

[3] When $\varphi_1 \neq n\pi$, that is, η + g is not real.

However, $\tan\varphi_2 = -\cot\varphi_1 \Rightarrow \varphi_2 = \varphi_1 + (2k+1)\frac{\pi}{2}$, and then $e^{2i\varphi_2} = e^{2i\varphi_1}e^{i(2k+1)\pi} = -e^{2i\varphi_1}$, for every $k = 0, 1, 2, \ldots$. Hence a general type-(5) spinor can be represented by

$$\psi_{(5)} = \begin{pmatrix} -\xi^* e^{2i\varphi_1} \\ \eta^* e^{2i\varphi_1} \\ \eta \\ \xi \end{pmatrix}. \tag{B7}$$

Writing $\psi_{(5)} = (\chi_2, \chi_1)^T$, it is straightforward to realize that $\chi_2 = -i\sigma_2\chi_1^* e^{2i\varphi_1} = \sigma_2\chi_1^* e^{i(2\varphi_1 - \frac{\pi}{2})}$. By taking $\varphi \equiv 2\varphi_1 - \frac{\pi}{2}$ a more compact form of (B7) is

$$\psi_{(5)} = \begin{pmatrix} e^{i\varphi}\sigma_2\chi_1^* \\ \chi_1 \end{pmatrix} \tag{B8}$$

By acting now the charge conjugation operator [23], with $i\Theta = \sigma_2$, it yields

$$C\psi_{(5)} = \mu\psi_{(5)}, \qquad \text{for} \quad C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K \qquad \text{and} \quad \mu = -e^{i\varphi}.$$

Here K conjugates the spinor components. Hence the eigenvalues take place on the sphere S^1. When these eigenvalues are real and χ_1, χ_2 are dual helicity eigenstates, Elko spinor fields are obtained. The rules to obtain type-(5) flagpole spinor fields from Dirac spinor fields are established in [28, 29] and shown to be Hopf fibrations [6].

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